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On the Rate of Convergence of Two Bernstein–Bézier Type Operators for Bounded Variation Functions*

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In this paper we study the rate of convergence of two Bernstein–Bézier type operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for bounded variation functions. By means of construction of suitable functions and the method of Bojanic and Vuillemier (*J. Approx. Theory* **31** (1981), 67–79), using some results of probability theory, we obtain asymptotically optimal estimations of $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for bounded variation functions at points of continuity and points of discontinuity. © 1998 Academic Press

1. INTRODUCTION

The object of this paper is to deal with approximation of bounded variation functions with two Bernstein–Bézier type operators. Let $P_{nk}(x) = \binom{n}{k} x^k(1-x)^{n-k}$ $(0 \leqslant k \leqslant n)$ be the Bernstein basis functions. Let $J_{nk}(x) = \sum_{j=k}^{n} P_{nj}(x)$ be the Bézier basis functions, which were introduced by P. Bézier [2]. For $\alpha \geqslant 1$, and a function f defined on $[0, 1]$, the Bernstein–Bézier operator $B_n^{(\alpha)}$ is defined by

$$B_n^{(\alpha)}(f, x) = \sum_{k=0}^{n} f(k/n)\, Q_{nk}^{(\alpha)}(x), \tag{1}$$

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and, for a function $f \in L_1[0, 1]$, the Bernstein–Kantorovich–Bézier operator $L_n^{(\alpha)}$ is defined by

$$L_n^{(\alpha)}(f, x) = (n+1) \sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x) \int_{I_k} f(t)\, \mathrm{d}t \tag{2}$$

where $I_k = [k/n+1, k+1/n+1]$ $(0 \leqslant k \leqslant n)$, and $Q_{nk}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $(J_{n,n+1}(x) \equiv 0)$. It is easily seen that $B^{(\alpha)}n$ and $L_n^{(\alpha)}$ are positive linear operators, $B_n^{(\alpha)}(1, x) = 1$, $L_n^{(\alpha)}(1, x) = 1$, and when $\alpha = 1$, the become the well-known Bernstein operator

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, P_{nk}(x), \tag{3}$$

and Bernstein–Kantorovich operator

$$L_n(f, x) = (n+1) \sum_{k=0}^{n} P_{nk}(x) \int_{I_k} f(t)\, \mathrm{d}t. \tag{4}$$

Let us first recall some results on the operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$, which are relevant to this paper. The Bernstein–Bézier operator $B_n^{(\alpha)}$ was defined by Chang [3] in 1983. Chang studied the convergence of $B_n^{(\alpha)}$ and showed that for $f \in C[0, 1]$, $\lim_{n \to +\infty} B_n^{(\alpha)}(f, x) = f(x)$ is uniform on $[0, 1]$. In 1985, Li and Gong [14] estimated the rate of convergence of $B_n^{(\alpha)}(f, x)$ for $f \in C[0, 1]$, and gave the following result.

Theorem A. *For $f \in C[0, 1]$, we have*

$$\|B_n^{(\alpha)}(f, x) - f(x)\|_{C[0,1]} \leqslant \begin{cases} (1+\alpha/4)\, \omega(n^{-1/2}, f), & \alpha \geqslant 1 \\ M\omega(n^{-\alpha/2}, f), & 0 < \alpha < 1 \end{cases}$$

where $\omega(\delta, f)$ is the modulus of continuity of $f(x)$. M is a constant depending only on α and f.

In 1986 Liu [11] obtained an inverse theorem for $B_n^{(\alpha)}$ in $C[0, 1]$ as follows:

Theorem B. *Let $f \in C[0, 1]$, $\alpha \geqslant 1$ and $0 < \beta < 1$ such that*

$$|B_n^{(\alpha)}(f, x) - f(x)| \leqslant M(\max\{n^{-1}, \sqrt{x(1-x)}/n^{1/2}\})^{\beta},$$

where M is a constant; then $f \in \mathrm{Lip}\,\beta$.

The Bernstein–Kantorovich–Bézier operator $L_n^{(\alpha)}(f)$, which is well adapted to approximation in the space $L_p(0,1)$, $1 \leqslant p < \infty$ was studied by Liu[5], who proved:

THEOREM C. *Let $1 \leqslant p < \infty$. Then we have for any $f \in L_p(0,1)$,*

$$\|L_n^{(\alpha)}(f) - f\|_p \leqslant M\omega(f, n^{-1/2})_p$$

where $\omega(f,t)_p$, is the modulus of continuity of f in $L_p[0,1]$, and M is a constant depending only on α and p.

In this paper, we shall consider the approximation of the operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ in the space BV[0, 1] and give asymptotically optimal estimates on the rate of convergence of $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for functions of bounded variation. We recall some results for Bernstein operator B_n, which are important for this paper.

Herzog and Hill [13] proved that if f is bounded on $[0,1]$ and x is a point of discontinuity of the first kind, then

$$\lim_{n\to\infty} B_n(f,x) = \tfrac{1}{2}(f(x+) + f(x-)). \tag{5}$$

Consequently, if f is a function of bounded variation on $[0,1]$, then (5) holds for all $x \in (0,1)$.

In 1983 Cheng [4] gave a rate of convergence of B_n for normalized bounded variation functions as follows:

Let f be of bounded variation on $[0,1]$. Then for every $x \in (0,1)$ and $n \geqslant (3/x(1-x))^8$, we have

$$\begin{aligned}&|B_n(f,x) - \tfrac{1}{2}(f(x+) + f(x-))| \\ &\quad\leqslant \frac{3}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) \\ &\quad\quad + \frac{18}{n^{1/6}(x(1-x))^{5/2}}\,|f(x+) - f(x-)|\end{aligned} \tag{6}$$

where $V_a^b(g_x)$ is total variation of g_x on $[a,b]$, and

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leqslant 1; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leqslant t < x. \end{cases}$$

In 1989 Guo and Khan [6] generalized Cheng's work by dealing with the approximation of Feller operators, which include some classical operators such as Bernstein, Szász, Baskakov, and Weierstrass operators.

In this paper, we shall improve and extend Cheng's work from another aspect by dealing with the approximation of two types of non-Feller operators, $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$, for bounded variation functions. ($B_n^{(\alpha)}$ and $L_n^{(\alpha)}$, except $B_n^{(1)}$, are not Feller operators, that is, $B_n^{(\alpha)}(t, x) \neq x$, and $L_n^{(\alpha)}(t, x) \neq x$; see 38, 40.) Furthermore, at the end of Section 2 we give an estimation for Bernstein operator B_n, which improves the result of Guo and Khan [6, (3.2)]. Our main results can be stated as follows:

THEOREM 1. *Let f be of bounded variation on $[0, 1]$. Then for every $x \in [0, 1]$ and $n \geqslant 1$, we have*

$$\left| B_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right|$$
$$\leqslant \frac{3\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{2\alpha}{\sqrt{nx(1-x)+1}} \left(|f(x+) - f(x-)| + e_n(x)\, |f(x) - f(x-)| \right) \tag{7}$$

where

$$e_n(x) = \begin{cases} 0, & \text{if } \quad x \neq k/n \quad \text{for all} \quad k \in N \\ 1, & \text{if } \quad x = k/n \quad \text{for a} \quad k \in N \end{cases}$$

(*when* $x = 0$, (*resp.*: $x = 1$), *we set* $(1/2^\alpha) f(x+) + (1 - (1/2^\alpha)) f(x-) = f(0)$ (*resp.*: $f(1)$)).

THEOREM 2. *Let f be of bounded variation on $[0, 1]$. Then for every $x \in (0, 1)$ and $n > 1/3x(1-x)$, we have*

$$\left| L_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right|$$
$$\leqslant \frac{5\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \frac{4\alpha}{\sqrt{nx(1-x)+1}} |f(x+) - f(x-)|. \tag{8}$$

In the particular case $\alpha = 1$, our Theorem 1 improves the result (6) of Cheng. Moreover, our results provide a more general case, since our bounded variation functions are not necessary normalized. In the last part, we shall prove that our estimates are asymptotically optimal for bounded variation functions at points of continuity and points of discontinuity.

Remark 1. The estimation term

$$\frac{2\alpha}{\sqrt{nx(1-x)}+1}\, e_n(x)\, |f(x)-f(x-)|$$

on the right of (7) is indispensable even when $\alpha=1$. The estimate (6) is incorrect without this term for unnormalized functions of bounded variation. For example, for the function

$$f_0(t)=\begin{cases}1, & t=\frac{1}{2}\\ 0, & 0\leqslant t<\frac{1}{2} \quad \text{or} \quad \frac{1}{2}<t\leqslant 1,\end{cases}$$

$f_0(t)$ is of bounded variation on $[0, 1]$. However estimate (6) is not true for $f_0(t)$, $n=2m$ positive even number, and $x=\frac{1}{2}$.

Using the Korovkin Theorem, we deduce from Theorems 1 and 2:

COROLLARY 1. *If $f(t)$ is bounded on $[0, 1]$, and if $x\in(0, 1)$ is a discontinuity point of the first kind of $f(t)$, we have*:

$$\lim_{n\to+\infty} B_n^{(\alpha)}(f, x)=\frac{1}{2^\alpha} f(x+)+\left(1-\frac{1}{2^\alpha}\right) f(x-) \tag{9}$$

and

$$\lim_{n\to+\infty} L_n^{(\alpha)}(f, x)=\frac{1}{2^\alpha} f(x+)+\left(1-\frac{1}{2^\alpha}\right) f(x-). \tag{10}$$

As $\alpha=1$, from (9) we obtain the result (5) of Herzog and Hill.
A part of our results has been announced in [7].

2. PROOF OF MAIN RESULTS

We will need the following lemmas for proving our results. Lemma 1 is the well-known Berry–Esseen bound for the classical central limit theorem of probability theory. Its proof and further discussion can be found in Loève [8, p. 300] and Feller [9, p. 515].

LEMMA 1. *Let $\{\xi_k\}_{k=1}^{+\infty}$ be a sequence of independent and identically distributed random variables with finite variance such that $E(\xi_1)=a_1\in R=(-\infty, +\infty)$ $\mathrm{Var}(\xi_1)=b_1^2>0$. Assume $E|\xi_1-E\xi_1|^3<\infty$, then there exists* *a numerical constant* C, $1/\sqrt{2\pi} \leqslant C < 0.8$, *such that for all* $n = 1, 2, ...,$ *and all* t,

$$\left| P\left(\frac{1}{b_1\sqrt{n}} \sum_{k=1}^{n} (\xi_k - a_1) \leqslant t\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du \right| < C \frac{E\,|\xi_1 - E\xi_1|^3}{\sqrt{n}\, b_1^3}. \tag{11}$$

LEMMA 2. *For* $x \in [0, 1]$, *we have*

$$\left| \sum_{nx < k \leqslant n} P_{nk}(x) - \frac{1}{2} \right| < \frac{0.8(2x^2 - 2x + 1) + 1/2}{\sqrt{nx(1-x)} + 1} \tag{12}$$

and

$$\left| \sum_{nx < k \leqslant n} P_{nk}(x) - \frac{1}{2} \right| < \frac{0.8(2x^2 - 2x + 1) + 1/6}{\sqrt{nx(1-x)} + 1/3} \tag{13}$$

Proof. Let ξ_1 be the random variable with two-point distribution $P(\xi_1 = j) = x^j (1-x)^{1-j}$ ($j = 0, 1,$ and $x \in [0, 1]$ is parameter), hence $a_1 = E\xi_1 = x$, $b_1 = \sqrt{x(1-x)}$, and $E\,|\xi_1 - E\xi_1|^3 = x(1-x)[x^2 + (1-x)^2]$. Let $\{\xi_k\}_{k=1}^{+\infty}$ be a sequence of independent random variables identically distributed with ξ_1, $\eta_n = \sum_{k=1}^{n} \xi_k$. Then the probability distribution of the random variable η_n is

$$P(\eta_n = k) = \binom{n}{k} x^k (1-x)^{n-k} = P_{nk}(x) \qquad (0 \leqslant k \leqslant n).$$

So

$$\sum_{nx < k \leqslant n} P_{nk}(x) = P(nx < \eta_n \leqslant n) = 1 - P(\eta_n \leqslant nx) = 1 - P\left(\frac{\eta_n - nx}{\sqrt{nx(1-x)}} \leqslant 0\right).$$

By Lemma 1, we get

$$\left| \sum_{nx < k \leqslant n} P_{nk}(x) - \frac{1}{2} \right| = \left| P\left(\frac{\eta_n - nx}{\sqrt{nx(1-x)}} \leqslant 0\right) - \frac{1}{2} \right|$$

$$< \frac{C}{\sqrt{n}} \frac{E\,|\xi_1 - E\xi_1|^3}{b_1^3} < \frac{0.8(2x^2 - 2x + 1)}{\sqrt{nx(1-x)}}$$

and noticing that $|\sum_{nx < k \leqslant n} P_{nk}(x) - \frac{1}{2}| \leqslant \frac{1}{2}$, (12), (13) are obtained.

From Lemma 2 we can prove that

$$\left| \sum_{(n+1)\,x<k\leqslant n} P_{nk}(x)-\frac{1}{2} \right| \leqslant \frac{0.8(2x^2-2x+1)+1}{\sqrt{nx(1-x)}+1}. \tag{14}$$

In fact, the interval $(nx, nx+x]$ includes an integer k' at most. By [10, Theorem 1], we have

$$P_{nk}(x) < \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nx(1-x)}}, \qquad \text{for} \quad 0 \leqslant k \leqslant n.$$

Hence, by Lemma 2

$$\begin{aligned} \left| \sum_{(n+1)\,x<k\leqslant n} P_{nk}(x)-\frac{1}{2} \right| &\leqslant \left| \sum_{nx<k\leqslant n} P_{nk}(x)-\frac{1}{2} \right| + P(nx<\eta_n \leqslant nx+x) \\ &\leqslant \frac{0.8(2x^2-2x+1)}{\sqrt{nx(1-x)}} + P_{nk'}(x) \\ &\leqslant \frac{0.8(2x^2-2x+1)}{\sqrt{nx(1-x)}} + \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nx(1-x)}}. \end{aligned}$$

Now (14) follows with the fact $|\sum_{(n+1)\,x<k\leqslant n} P_{nk}(x)-\frac{1}{2}| \leqslant \frac{1}{2}$.

LEMMA 3. *For $x \in [0, 1]$, there holds*

$$Q_{nk}^{(\alpha)}(x) < \frac{3}{2} \frac{\alpha}{\sqrt{nx(1-x)}+1} \tag{15}$$

and

$$Q_{nk}^{(\alpha)}(x) < \frac{5}{6} \frac{\alpha}{\sqrt{nx(1-x)}+1/3}. \tag{16}$$

Proof. It is easy to verify that $|a^\alpha - b^\alpha| \leqslant \alpha\, |a-b|$ $(0 \leqslant a,\ b \leqslant 1,\ \alpha \geqslant 1)$, and by [10, Theorem 1], we obtain

$$Q_{nk}^{(\alpha)}(x) \leqslant \alpha P_{nk}(x) < \frac{1}{\sqrt{2e}} \frac{\alpha}{\sqrt{nx(1-x)}}.$$

Since $Q_{nk}^{(\alpha)}(x) \leqslant 1$, (15), (16) are proved.

Now we define the function $\widetilde{\operatorname{sgn}}(t)$ by

$$\widetilde{\operatorname{sgn}}(t)=\begin{cases}2^\alpha-1, & t>0;\\ 0, & t=0;\\ -1, & t<0.\end{cases}$$

Estimates of $B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x)$ and $L_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x)$ are important for proving our theorems.

LEMMA 4. *We have*

$$B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x)=2^\alpha\left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha-1+e_n(x)\,Q_{nk}^{(\alpha)}(x). \tag{17}$$

Proof. Since

$$B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x)=(2^\alpha-1)\sum_{nx<k\leqslant n} Q_{nk}^{(\alpha)}(x)-\sum_{0\leqslant k<nx} Q_{nk}^{(\alpha)}(x)$$

and

$$1=B_n^{(\alpha)}(1, x)=\sum_{nx<k\leqslant n} Q_{nk}^{(\alpha)}(x)+\sum_{0\leqslant k<nx} Q_{nk}^{(\alpha)}(x)+e_n(x)\,Q_{nk}^{(\alpha)}(x),$$

it follows that

$$\begin{aligned}B_n^{(\alpha)}&(\widetilde{\operatorname{sgn}}(t-x), x)\\ &=(2^\alpha-1)\sum_{nx<k\leqslant n} Q_{nk}^{(\alpha)}(x)-\left[1-\sum_{nx<k\leqslant n} Q_{nk}^{(\alpha)}(x)-e_n(x)\,Q_{nk}^{(\alpha)}(x)\right]\\ &=2^\alpha\left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha-1+e_n(x)\,Q_{nk}^{(\alpha)}(x).\end{aligned}$$

LEMMA 5. *We have*

$$\begin{aligned}&\left|\frac{f(x+)-f(x-)}{2^\alpha}B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x)\right.\\ &\qquad\left.+\left[f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right]B_n^{(\alpha)}(\delta_x, x)\right|\\ &\quad\leqslant\frac{2\alpha}{\sqrt{nx(1-x)+1}}\left(|f(x+)-f(x-)|+e_n(x)\,|f(x)-f(x-)|\right)\end{aligned} \tag{18}$$

and

$$\left|\frac{f(x+)-f(x-)}{2^\alpha} B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x) + \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1-\frac{1}{2^\alpha}\right) f(x-)\right] B_n^{(\alpha)}(\delta_x, x)\right|$$
$$\leqslant \alpha \frac{0.8(2x^2-2x+1)+1/6}{\sqrt{nx(1-x)}+1/3} |f(x+)-f(x-)| + \frac{5}{6} \frac{\alpha e_n(x) |f(x)-f(x-)|}{\sqrt{nx(1-x)}+1/3}, \tag{19}$$

where

$$\delta_x(t) = \begin{cases} 0, & t \neq x; \\ 1, & t = x. \end{cases}$$

Proof. We have $B_n^{(\alpha)}(\delta_x, x) = e_n(x)\, Q_{nk}^{(\alpha)}(x)$. By (12), (15), and Lemma 4 we get

$$\left|\frac{f(x+)-f(x-)}{2^\alpha} B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x) + \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1-\frac{1}{2^\alpha}\right) f(x-)\right] B_n^{(\alpha)}(\delta_x, x)\right|$$
$$= \left|\frac{f(x+)-f(x-)}{2^\alpha}\left[2^\alpha \left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha - 1\right] + [f(x)-f(x-)]\, e_n(x)\, Q_{nk}^{(\alpha)}(x)\right|$$
$$\leqslant \frac{4}{3}\frac{\alpha}{\sqrt{nx(1-x)}+1} |f(x+)-f(x-)| + \frac{3}{2}\frac{\alpha}{\sqrt{nx(1-x)}+1} e_n(x)\, |f(x)-f(x-)|$$
$$\leqslant \frac{2\alpha}{\sqrt{nx(1-x)}+1} (|f(x+)-f(x-)| + e_n(x)\, |f(x)-f(x-)|),$$

and from (13), (16), and (17), we get (19).

LEMMA 6. *The following inequality holds*:

$$\left|\frac{f(x+)-f(x-)}{2^\alpha} L_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x)\right| < \frac{4\alpha}{\sqrt{nx(1-x)}+1} |f(x+)-f(x-)|. \tag{20}$$

Proof. Let $x \in [k'/n+1, k'+1/n+1)$; then

$$\begin{aligned} L_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x) &= (n+1) \sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x) \int_{I_k} \widetilde{\operatorname{sgn}}(t-x)\, \mathrm{d}t \\ &= 2^\alpha \sum_{k=k'}^{n} Q_{nk}^{(\alpha)}(x) - 1 + 2^\alpha Q_{nk'}^{(\alpha)}(x)(k'-nx-x) \\ &\leqslant \left|2^\alpha \left(\sum_{(n+1)\,x<k\leqslant n} P_{nk}(x)\right)^\alpha - 1\right| + 2^\alpha Q_{nk'}^{(\alpha)}(x). \end{aligned}$$

Using (14) and Lemma 3 we get (20).

Let

$$K_{n,\alpha}^{(1)}(x, t) = \begin{cases} \sum_{k\leqslant nt} Q_{nk}^{(\alpha)}(x), & 0<t\leqslant 1 \\ 0, & t=0 \end{cases}$$

and

$$K_{n,\alpha}^{(2)}(x, t) = \sum_{k=0}^{n} (n+1)\, Q_{nk}^{(\alpha)}(x)\, \chi_k(t),$$

where χ_k is the characteristic function of the interval I_k with respect to $I=[0, 1]$.

We recall the Lesbesgue–Stieltjes integral representations:

$$B_n^{(\alpha)}(f, x) = \int_0^1 f(t)\, \mathrm{d}_t K_{n,\alpha}^{(1)}(x, t) \tag{21}$$

and

$$L_n^{(\alpha)}(f, x) = \int_0^1 f(t)\, K_{n,\alpha}^{(2)}(x, t)\, \mathrm{d}t. \tag{22}$$

LEMMA 7. *For every* $x \in (0, 1)$, *and* $n > 1/3x(1-x)$, *we have*

$$L_n^{(1)}((t-x)^2, x) < \frac{2x(1-x)}{n}. \tag{23}$$

Proof. After simple calculation, we get

$$L_n^{(1)}(1, x) = 1,$$

$$L_n^{(1)}(t, x) = x + \frac{1-2x}{2(n+1)},$$

$$L_n^{(1)}(t^2, x) = x^2 + \frac{x(2-3x)}{n+1} + \frac{1-6x+6x^2}{3(n+1)^2}.$$

Hence, as $n > 1/3x(1-x)$, there holds

$$L_n^{(1)}((t-x)^2, x) = \frac{x(1-x)}{n+1} + \frac{1-6x+6x^2}{3(n+1)^2} < \frac{2x(1-x)}{n}.$$

LEMMA 8. *Let* $x \in (0, 1)$; *then*

(i) *For* $0 \leqslant y < x$, *we have* $K_{n,\alpha}^{(1)}(x, y) \leqslant \alpha x(1-x)/n(x-y)^2$.

(ii) *For* $x < z \leqslant 1$, *we have* $1 - K_{n,\alpha}^{(1)}(x, z) \leqslant \alpha x(1-x)/n(x-z)^2$.

Proof. Let

$$K_n(x, t) = \begin{cases} \sum_{k \leqslant nt} P_{nk}(x), & \text{if } \ 0 < t \leqslant 1; \\ 0, & \text{if } \ t = 0. \end{cases}$$

Then

$$K_{n,\alpha}^{(1)}(x, y) \leqslant \alpha K_n(x, y) = \alpha \int_0^y \mathrm{d}_t K_n(x, t) \leqslant \alpha \int_0^y \left(\frac{x-t}{x-y}\right)^2 \mathrm{d}_t K_n(x, t)$$

$$\leqslant \frac{\alpha}{(x-y)^2} \int_0^1 (x-t)^2 \, \mathrm{d}_t K_n(x, t) = \alpha \frac{x(1-x)}{n(x-y)^2}. \tag{24}$$

Similarly (ii) is proved.

Using a similar method we prove

LEMMA 9. *For* $x \in (0, 1)$, *if* $n > 1/3x(1-x)$, *then*

(i) *For* $0 \leqslant y < x$, *we have* $\int_0^y K_{n,\alpha}^{(2)}(x, t)\, \mathrm{d}t \leqslant 2\alpha x(1-x)/n(x-y)^2$.

(ii) *For* $x < z \leqslant 1$, *we have* $\int_z^1 K_{n,\alpha}^{(2)}(x, t)\, \mathrm{d}t \leqslant 2\alpha x(1-x)/n(x-z)^2$.

Now we need estimates of $B_n^{(\alpha)}(g_x, x)$ and $L_n^{(\alpha)}(g_x, x)$.

Lemma 10. *For $n \geqslant 1$, we have*

$$|B_n^{(\alpha)}(g_x, x)| \leqslant \frac{3\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{25}$$

Proof. The proof of this lemma is based on the method of Bojanic and Vuillemier [1] (see also Cheng [4]). We decompose $[0, 1]$ into three parts:

$$I_1^* = [0, x - x/\sqrt{n}], \qquad I_2^* = [x - x/\sqrt{n}, x + (1-x)/\sqrt{n}],$$
$$I^*{}_3^* = [x + (1-x)/\sqrt{n}, 1].$$

By (21), we have

$$B_n^{(\alpha)}(g_x, x) = \int_0^1 g_x(t)\, d_t K_{n,\alpha}^{(1)}(x, t) = \Delta_{1,n}(f, x) + \Delta_{2,n}(f, x) + \Delta_{3,n}(f, x)$$

where

$$\Delta_{j,n}(f, x) = \int_{I_j^*} g_x(t)\, d_t K_{n,\alpha}^{(1)}(x, t) \qquad \text{for} \quad j = 1, 2, 3.$$

For $t \in I_2^*$, we have $|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x)$.
So

$$|\Delta_{2,n}(f, x)| \leqslant V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x) \leqslant \frac{1}{n-1} \sum_{k=2}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{26}$$

To estimate $\Delta_{1,n}(f, x)$, let $y = x - x/\sqrt{n}$. Using partial integration and Lemma 8, we get

$$\begin{aligned}
|\Delta_{1,n}(f, x)| &= \left| \int_0^y g_x(t)\, d_t K_{n,\alpha}^{(1)}(x, t) \right| \\
&= \left| g_x(y+)\, K_{n,\alpha}^{(1)}(x, y) - \int_0^y K_{n,\alpha}^{(1)}(x, t)\, d_t g_x(t) \right| \\
&\leqslant V_{y+}^x(g_x)\, K_{n,\alpha}^{(1)}(x, y) + \int_0^y K_{n,\alpha}^{(1)}(x, t)\, d_t(-V_t^x(g_x)) \\
&\leqslant V_{y+}^x(g_x) \frac{\alpha x(1-x)}{n(x-y)^2} + \frac{\alpha x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)).
\end{aligned}$$

Since

$$\int_0^y \frac{1}{(x-t)^2}\, \mathrm{d}_t(-V_t^x(g_x)) = -\frac{1}{(x-t)^2}\, V_t^x(g_x)|_0^{y+} + \int_0^y V_t^x(g_x)\, \frac{2}{(x-t)^3}\, \mathrm{d}t$$

we have

$$|\Delta_{1,n}(f,x)| \leqslant \frac{\alpha x(1-x)}{nx^2}\, V_0^x(g_x) + \frac{\alpha x(1-x)}{n} \int_0^y V_t^x(g_x)\, \frac{2}{(x-t)^3}\, \mathrm{d}t.$$

Putting $t = x - x/\sqrt{u}$ for the last integral, we get

$$\int_0^y V_t^x(g_x)\, \frac{2}{(x-t)^3}\, \mathrm{d}t = \frac{1}{x^2} \int_1^n V_{x-x/\sqrt{u}}^x(g_x)\, \mathrm{d}u \leqslant \frac{1}{x^2} \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x).$$

Consequently

$$|\Delta_{1,n}(f,x)| \leqslant \frac{\alpha x(1-x)}{nx^2}\, V_0^x(g_x) + \frac{\alpha x(1-x)}{nx^2} \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \tag{27}$$

Using the same method for estimating $|\Delta_{3,n}(f,x)|$, we have

$$|\Delta_{3,n}(f,x)| \leqslant \frac{\alpha x(1-x)}{n(1-x)^2}\, V_x^1(g_x) + \frac{\alpha x(1-x)}{n(1-x)^2} \sum_{k=1}^n V_x^{x+(1-x)/\sqrt{k}}(g_x). \tag{28}$$

From (25), (27), (28), it follows that

$$|B_n^{(\alpha)}(g_x,x)| \leqslant \frac{2\alpha}{nx(1-x)} \sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x).$$

On the other hand, noticing that $|B_n^{(\alpha)}(g_x,x)| \leqslant V_0^1(g_x)$, we get (25).

In the same manner, we can prove

LEMMA 11. *For every* $x \in (0,1)$, *and* $n > 1/3x(1-x)$, *we have*

$$|L_n^{(\alpha)}(g_x,x)| \leqslant \frac{5\alpha}{nx(1-x)+1} \sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{29}$$

Now we prove Theorems 1 and 2. Note that for all t

$$\begin{aligned} f(t) = {} & \frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2^\alpha}\, \widetilde{\operatorname{sgn}}(t-x) \\ & + \delta_x(t) \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right], \end{aligned} \tag{30}$$

and $B_n^{(\alpha)}(\delta_x, x) = e_n(x)\, Q_{nk}^{(\alpha)}(x)$, $L_n^{(\alpha)}(\delta_x, x) = 0$, we get

$$\begin{aligned}
&\left| B_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right| \\
&\quad \leqslant |B_n^{(\alpha)}(g_x, x)| + \left| \frac{f(x+) - f(x-)}{2^\alpha} B_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x) \right. \\
&\quad\quad \left. + \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] B_n^{(\alpha)}(\delta_x, x) \right|, \qquad (31)
\end{aligned}$$

and

$$\begin{aligned}
&\left| L_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right| \\
&\quad \leqslant |L_n^{(\alpha)}(g_x, x)| + \left| \frac{f(x+) - f(x-)}{2^\alpha} L_n^{(\alpha)}(\widetilde{\operatorname{sgn}}(t-x), x) \right|. \qquad (32)
\end{aligned}$$

By Lemmas 5 and 10, we obtain Theorem 1; and by Lemmas 6 and 11, we obtain Theorem 2.

To prove Corollary 1, we deduce from Theorem 1 that

$$\lim_{n \to +\infty} B_n^{(\alpha)}((t-x)^2, x) = 0.$$

Noticing that $g_x(t)$ at point x is continuous, by the well-known Korovkin Theorem, we have

$$\lim_{n \to +\infty} B_n^{(\alpha)}(g_x, x) = 0$$

and by Lemma 5, the right member of (31) tends to $0 (n \to +\infty)$, and (9) is proved.

The proof of (10) is analogous.

We have proved Theorems 1 and 2 and Corollary 1. We conclude this section by giving an immediate consequence of (30), (19), and (25):

PROPOSITION 1. *Let f be of bounded variation on $[0, 1]$. Then for every $x \in [0, 1]$ such that $f(x) = (f(x+) + f(x-))/2$ and every $n \geqslant 1$, we have*

$$\left| B_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] \right|$$
$$\leqslant \frac{3\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$$
$$+ \left[0.8(2x^2-2x+1) + 1/6 + \frac{5}{12} e_n(x) \right]$$
$$\times \frac{\alpha}{\sqrt{nx(1-x)+1/3}} |f(x+)-f(x-)|$$

(when $x=0$, *(resp.:* $x=1$), *we set* $(1/2^\alpha) f(x+) + (1-(1/2^\alpha)) f(x-) = f(0)$, *(resp.:* $f(1)$).

In the particular case $\alpha = 1$, our Proposition 1 improves a result of Guo and Khan [6, (3.2)].

3. OUR ESTIMATES ARE SHARP

We shall show that our estimates are asymptotically optimal when $n \to +\infty$. First we need the following lemmas:

LEMMA 12. *For* $x \in (0, 1)$, $f(t)$: $|t-x|$; *then for* $n > 256/x(1-x)$, *we have*

$$L_n^{(1)}(|(t-x)|, x) \geqslant \frac{1}{32} \left(\frac{x(1-x)}{n} \right)^{1/2}. \tag{33}$$

Proof. Let $B_n(f, x) = \sum_{k=0}^{n} f(k/n) P_{nk}(x)$ be the Bernstein operator; then

$$B_n(|t-x|, x) - L_n^{(1)}(|t-x|, x)$$
$$\leqslant (n+1) \sum_{k=0}^{n} P_{nk}(x) \int_{I_k} \big| |t-x| - |k/n - x| \big| \, dt$$
$$\leqslant (n+1) \sum_{k=0}^{n} P_{nk}(x) \int_{I_k} |t - k/n| \, dt$$
$$\leqslant \frac{1}{2n+2} \sum_{k=0}^{n} P_{nk}(x) < \frac{1}{2n}.$$

By [4, (2.6)], as $n > 256/x(1-x)$, there holds

$$L_n^{(1)}(|t-x|, x) \geqslant B_n(|t-x|, x) - \frac{1}{2n} \geqslant \frac{1}{16}\left(\frac{x(1-x)}{n}\right)^{1/2} - \frac{1}{2n}$$
$$\geqslant \frac{1}{32}\left(\frac{x(1-x)}{n}\right)^{1/2}.$$

LEMMA 13. *For n a positive even number $n = 2m$ ($m = 1, 2, 3, \ldots$), we have*

$$\frac{1}{2^\alpha} - J^\alpha_{2m,\, m+1}\left(\frac{1}{2}\right) > \left(\frac{1}{4}\right)^\alpha \frac{1}{\sqrt{n}}. \tag{34}$$

Proof. Obviously,

$$\sum_{k=0}^{2m} P_{2m,\,k}(\tfrac{1}{2}) = 1, \qquad \text{and} \qquad \sum_{k=0}^{m-1} P_{2m,\,k}(\tfrac{1}{2}) = \sum_{k=m+1}^{2m} P_{2m,\,k}(\tfrac{1}{2}).$$

So

$$J_{2m,\,m+1}(\tfrac{1}{2}) = \tfrac{1}{2} - \tfrac{1}{2} P_{2m,\,m}(\tfrac{1}{2}) \geqslant \tfrac{1}{4}.$$

Using Stirling's formula $n! = (2\pi n)^{1/2} n^n e^{-n} e^{\theta n/12n}$, where $0 < \theta_n < 1$, we obtain

$$P_{2m,\,m}\left(\frac{1}{2}\right) = \frac{(2m)!}{(m!)^2}\left(\frac{1}{2}\right)^{2m} = \frac{e^{\theta_{2m}/24m}}{\sqrt{\pi m}\; e^{\theta m/6m}}.$$

So we have

$$\frac{1}{2}\frac{1}{\sqrt{n}} < \sqrt{2/\pi}\; e^{-1/6}\frac{1}{\sqrt{n}} < P_{2m,\,n}\left(\frac{1}{2}\right) < \sqrt{2/\pi}\; e^{1/24}\frac{1}{\sqrt{n}}. \tag{35}$$

By the mean-value theorem and the above inequality for $J_{2m,\,m+1}(\frac{1}{2})$, it follows that

$$\left|\frac{1}{2^\alpha} - J^\alpha_{2m,\,m+1}\left(\frac{1}{2}\right)\right| = \frac{1}{2^\alpha} - J^\alpha_{2m,\,m+1}\left(\frac{1}{2}\right) = \alpha\gamma_m^{\alpha-1}\frac{1}{2} P_{2m,\,m}\left(\frac{1}{2}\right)$$

where $1/4 \leqslant J_{2m,\,m+1}(\frac{1}{2}) < \gamma_m < \frac{1}{2}$.

From (35), we have

$$\alpha\gamma_m^{\alpha-1}\frac{1}{2} P_{2m,\,m}\left(\frac{1}{2}\right) > \alpha\left(\frac{1}{4}\right)^{\alpha-1}\frac{1}{2}\cdot\frac{1}{2}\frac{1}{\sqrt{n}} > \left(\frac{1}{4}\right)^\alpha \frac{1}{\sqrt{n}}.$$

Now we prove that our estimates (7) and (8) are asymptotically optimal for continuity points and discontinuity points of bounded variation function $f(t)$. If x is the continuity point of f, (7) and (8) become

$$|B_n^{(\alpha)}(f, x) - f(x)| \leqslant \frac{3\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(f) \tag{36}$$

and

$$|L_n^{(\alpha)}(f, x) - f(x)| \leqslant \frac{5\alpha}{nx(1-x)+1} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(f). \tag{37}$$

For $\alpha \neq 1$, consider $f(t) = t$; from (36) and [11, Lemma 4], when n is sufficiently large, we have

$$C_1 \frac{\sqrt{x(1-x)}}{\sqrt{n}} \leqslant |B_n^{(\alpha)}(t, x) - x| \leqslant \frac{6\alpha}{\sqrt{n}\, x(1-x) + 1/\sqrt{n}} \tag{38}$$

where C_1 is a positive constant and

$$L_n^{(\alpha)}(t, x) - x = \left(\sum_{k=0}^{n} (k/n)\, Q_{nk}^{(\alpha)}(x) - x \right) + \frac{1}{n+1} \left(\frac{1}{2} - \sum_{k=0}^{n} (k/n)\, Q_{nk}^{(\alpha)}(x) \right). \tag{39}$$

Hence from (37) and a simple comparison between $L_n^{(\alpha)}(t, x)$ and $B_n^{(\alpha)}(t, x)$, when n is sufficiently large, we have

$$C_2 \frac{\sqrt{x(1-x)}}{\sqrt{n}} \leqslant |L_n^{(\alpha)}(t, x) - x| \leqslant \frac{10\alpha}{\sqrt{n}\, x(1-x) + 1/\sqrt{n}} \tag{40}$$

where C_2 is a positive constant.

When $\alpha = 1$, for $B_n^{(1)}$ the conclusion is known (see [4]). For $L_n^{(1)}$, we take $f(t) = |t - x|$ $(0 < x < 1)$, by (33) and (37), if $n > 256/x(1-x)$, we have

$$\frac{1}{32} \left(\frac{x(1-x)}{n} \right)^{1/2} \leqslant |L_n^{(1)}(|t-x|, x)| \leqslant \frac{10}{\sqrt{n}\, x(1-x) + 1/\sqrt{n}}. \tag{41}$$

From (38), (40, (41), we deduce that (36) and (37) cannot be asymptotically improved when $n \to +\infty$.

For the discontinuity point of f, when $g_x \equiv 0$, (7) and (8) become

$$\left|B_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-)\right]\right| \leqslant \frac{2\alpha}{\sqrt{nx(1-x)}+1} \left(|f(x+) - f(x-)| + e_n(x)\, |f(x) - f(x-)|\right) \tag{42}$$

and

$$\left|L_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2\alpha}\right) f(x-)\right]\right| \leqslant \frac{4\alpha}{\sqrt{nx(1-x)}+1} |f(x+) - f(x-)|. \tag{43}$$

We take

$$f(t) = \begin{cases} 1, & 0 \leqslant t \leqslant \frac{1}{2} \\ 0, & \frac{1}{2} < t \leqslant 1 \end{cases}$$

and $x = \frac{1}{2}$, $n = 2m$. Now $g_x(t) \equiv 0$.

By Lemma 13 and (42), we get

$$\left(\frac{1}{4}\right)^\alpha \frac{1}{\sqrt{n}} \leqslant \left|B_n^{(\alpha)}(f, x) - \left[\frac{1}{2^\alpha} f(x+) + \left(1 - \frac{1}{2^\alpha}\right) f(x-)\right]\right| \leqslant \frac{4\alpha}{\sqrt{n}}. \tag{44}$$

Therefore, (42) cannot be asymptotically improved when $n \to +\infty$, and the proof of the same property for (43) is similar.

Remark 2. Some other classical operators, such as those of Szász, Baskakov, and Meyerkönig and Zeller, can be modified in a way similar to that for the Bernstein operator. The methods for approximation of these modified operators are different. We shall discuss them elsewhere.

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